AFCRL 62-267

GCA TECHNICAL REPORT NO. 62-3-A

GEOPHYSICS CORPORATION OF AMERICA BEDFORD, MASSACHUSETTS

THE PENETRATING POWER OF PARTICULATE MATTER IN AN EXPONENTIAL ATMOSPHERE

H. K. BROWN, J. PRESSMAN AND F. F. MARMO

"THIS RESEARCH IS A PART OF PROJECT DEFENDER SPONSORED BY THE ADVANCED RESEARCH PROJECTS AGENCY, DEPARTMENT OF DEFENSE"

ARPA ORDER NO. 258 - 61 PROJECT 5842

SCIENTIFIC REPORT NO. 3

CONTRACT NO. AF 19(628)-231

PREPARED FOR

GEOPHYSICS RESEARCH DIRECTORATE

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES

OFFICE OF AEROSPACE RESEARCH

UNITED STATES AIR FORCE, BEDFORD, MASSACHUSETTS

FEBRUARY 1962

Requests for additional copies by Agencies of the Department of Defense, their contractors, and other Government agencies should be directed to the:

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

Department of Defense contractors must be established for ASTIA services or have their "need-to-know" certified by the cognizant military agency of their project or contract.

All other persons and organizations should apply to the:

U. S. DEPARTMENT OF COMMERCE
OFFICE OF TECHNICAL SERVICES
WASHINGTON 25, D. C.

AFCRL - 62-267

THE PENETRATING POWER OF PARTICULATE MATTER IN AN EXPONENTIAL ATMOSPHERE

H. K. Brown

J. Pressman

F. F. Marmo

SCIENTIFIC REPORT NO. 3
Contract No. AF 19(628)-231

ARPA Order No. 258-61

GCA Technical Report No. 62-3-A
Project No. 5842

February 1962

GEOPHYSICS CORPORATION OF AMERICA
Bedford, Massachusetts

This research is a part of Project DEFENDER
Sponsored by the Advanced Research Project Agency, Department of Defense

Prepared for

GEOPHYSICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

TABLE OF CONTENTS

INTRODUCTION

The light emitted from missile trails represents an exceedingly complex phenomenon. A large variety of physical theories have been offered to explain this luminosity. Since particulate matter is generally found to some degree in missile exhausts an understanding of the interaction of particulate matter with the atmosphere may provide a key to the disentanglement of the particulate matter contribution. The thesis here is that if a particular zone of luminsoity is observed to move in a certain fashion which is inappropriate to the movement of particulate matter then the invoking of the particulate matter mechanism cannot be accepted.

In an effort to provide a means for checking the particulate matter hypothesis this study has been made. It attempts to systematically obtain the mathematical formulae for the velocity and distance behavior of particles of different mass and initial velocity ejected at various heights in an exponential atmosphere. Its purpose is essentially that of an engineering type study for application to the above cited missile phenomena. Hence for clarification specific examples are calculated in detail for the various cases. Also a more extensive tabulation for horizontal motion with drag is given in an appendix.

By use of the work contained herein it is hoped that the particulate matter component in missile trail luminsoity may be more carefully checked. This study represents a small effort in a complex over-all program of evaluating the missile trail radiation.

If particulate matter is released in the upper atmosphere, at a point, with a common initial speed in all directions, the drag resistance of the atmosphere will slow down the particles in such a manner that the space loci of the advancing particles, with a common varying speed, will describe successively expanding aspherical surfaces. That is to say, the surfaces of constant speed will be strongly dependent upon the varying density, $\rho(h)$, of the upper atmosphere, which is essentially exponential,

$$\rho = \rho(h) = \rho_o \exp(-h/H_o) \tag{1}$$

for different scale heights H_o.

The aerodynamic drag, D, on a particle of speed v and resisting area A, moving through such a density field, is defined, in terms of a drag coefficient C_D, by the standard formula

$$D = \frac{1}{2} C_D A \rho v^2 \tag{2}$$

acting in such a manner as to directly oppose the motion of the particle. The force of gravity can be taken to be constant, $W = mg$, in a downward direction.

The combined action of aerodynamic drag and gravity on the particle is to impart to the particle an acceleration $\bar{a}$ defined by the Newtonian equation of motion

$$m\bar{a} = - D \frac{\bar{v}}{v} - W \bar{e}_h \tag{3}$$

where the velocity vector $\bar{v}$ is defined by the equations

$$\bar{a} = d\bar{v}/dt \tag{4}$$

and

$$\bar{v} = \frac{dr}{dt} \bar{e}_r + \frac{dh}{dt} \bar{e}_h \tag{5}$$

where $\bar{e}_h$ is a unit vector in the upward direction and $\bar{e}_r$ is a unit vector in the radial direction, out from the axis of symmetry, the vertical axis through the initial release point $r = 0$, $h = h_o$.

Equation (3) is equivalent to the two componential equations of motion

$$m \frac{d^2r}{dt^2} = - \frac{D}{v} \frac{dr}{dt} \tag{6a}$$

and

$$m \frac{d^2h}{dt^2} = - \frac{D}{v} \frac{dh}{dt} - W \tag{6b}$$

where

$$v^2 = (dr/dt)^2 + (dh/dt)^2 \tag{7}$$

The initial conditions of motion, when $t = 0$, are defined by the statements:

$$\left.\begin{aligned} &r = 0, \qquad dr/dt = v_o \sin \varphi_o, \\ &h = h_o, \qquad dh/dt = v_o \cos \varphi_o, \qquad 0 \leqq \varphi_o < \pi \end{aligned}\right\} \tag{8}$$

the parameter φ_o selects a specified set of particles issuing from the apex of a cone of generating angle φ_o, measured from the vertical line through $r = 0$, $h = h_o$.

If the times, t, of observation are very short so that the quantity gt is small compared to v_o, one may neglect the gravitational action in describing the motion of the particles. In this event, with no gravity, there are three principal cases of motion with the following differential equations and initial conditions:

Case I: Horizontal Motion with Drag

$$dv_H/dt = - C\, v_H^2 \tag{9a}$$

$$v_H = ds_H/dt \tag{9b}$$

$$v_H = v_o, \qquad s_H = 0, \qquad \text{when } t = 0 \tag{9c}$$

where

$$C = \frac{1}{2} (C_D A/m)\, \rho_o \exp(-h_o/H_o) \tag{10}$$

Case II: Upward Motion with Drag

$$dv_U/dt = - C \exp(-s_U/H_o) v_U^2 \tag{11a}$$

$$v_U = ds_U/dt, \qquad s_U = h - h_o \tag{11b}$$

$$v_U = v_o, \qquad s_U = 0, \qquad \text{when } t = 0 \tag{11c}$$

Case III: Downward Motion with Drag

$$dv_D/dt = - C \exp(s_D/H_o) v_D^2 \tag{12a}$$

$$v_D = ds_D/dt, \qquad s_D = h_o - h \tag{12b}$$

$$v_D = v_o, \qquad s_D = 0, \qquad \text{when } t = 0 \tag{12c}$$

When a constant gravity field is introduced into the description of motion, as it should be, two additional cases may be added to our schedule:

Case IV: Upward Motion with Drag and Gravity

$$dv_V/dt = - C \exp(-s_V/H_o) v_V^2 - g \tag{13a}$$

$$v_V = ds_V/dt, \qquad s_V = h - h_o \tag{13b}$$

$$v_V = 0, \qquad s_V = 0, \qquad \text{when } t = 0 \tag{13c}$$

Case V: Downward Motion with Drag and Gravity

$$dv_{\ell}/dt = - C \exp(s_{\ell}/H_o) + g \tag{14a}$$

$$v_{\ell} = ds\ /dt, \qquad s_{\ell} = h_o - h \tag{14b}$$

$$v_{\ell} = v_o, \qquad s_{\ell} = 0, \qquad \text{when } t = 0 \tag{14c}$$

An analysis of these five cases is made in this report. The sixth case of horizontal motion (initial release, that is) with drag and gravity is not discussed, nor is the general problem, described by Equation (3), discussed. In order to illustrate the use of the formulae, as they are developed in each successive case, application is made to a release of particles at 71 Km, with an initial speed v_o = 2500 cm/sec. The numerical examples serve as a means of comparing the various features and salient points of each case.

In the formation and interpretation of missile trails, it is pertinent to know at what altitude one can expect a specified drop of initial velocity in a specified time, and to know the penetrating distances s_H, s_U, and s_D. In the Appendix, a table of results is given for a given set of initial velocities in the horizontal direction, for Case I.

In closing, it is noted that this paper is an introductory engineering study only, revealing points and topics worthy of further study. The fact that the illustrations used are couched in the language of missile trails should not obscure the evident fact of the generality of this study and its applicability to the solution of problems of the relative motion of bodies in an exponential atmosphere, with gravity.

I. HORIZONTAL MOTION WITH DRAG

In this section we consider Case I, motion in a horizontal direction with drag only. The dynamic equation of motion

$$dv_H/dt = - C\, v_H^{\,2}, \qquad v_H = ds_H/dt \tag{9a}$$

with the initial condition

$$v_H = v_o, \qquad \text{when } t = 0 \tag{9c}$$

by direct integration yields the velocity formula

$$(1/v_H) - (1/v_o) = C\, t_H \tag{15}$$

which can be written in the forms

$$v_o/v_H = 1 + C\, v_o\, t_H \tag{16a}$$

$$C = \frac{1}{v_o t_H}\left(\frac{v_o}{v_H} - 1 \right) \tag{16b}$$

On the other hand, since $v_H = ds_H/d$' Equation (9a) can be written as

$$dv_H/dt = - C\, v_H\, (ds_H/dt) \tag{17}$$

which has the integral

$$v_H = v_o \exp(-C\, s_H) \tag{18}$$

since $v_H = 0$, when $s_H = 0$. That is to say,

$$v_o/v_H = \exp(C\ s_H) \tag{19}$$

so that

$$s_H = \frac{1}{C}\ \ell n(v_o/v_H) \quad . \tag{20}$$

By using

$$\frac{1}{C} = \frac{v_o t_H}{(v_o/v_H) - 1} \tag{21}$$

from Equation (16), we can express s_H in the significant form

$$s_H = \left[\frac{\ell n(v_o/v_H)}{(v_o/v_H) - 1}\right]\ v_o t_H = \frac{1}{C}\ \ell n(1 + C\ v_o t_H) \quad . \tag{22}$$

<u>Equation (22) defines a formula for the horizontal displacement, s_H, of the particle in the time $t = t_H$, when the velocity ration is v_H/v_o.</u>

It is interesting to note that, for a fixed value of v_H/v_o, this formula for s_H states that the displacement s_H is proportional to $v_o t$, the displacement without resistance, when $C = 0$ (i.e., when $C_D \equiv 0$, say). Hence, Equation (22) can be written as

$$\frac{s_H}{s_{H_{C=0}}} = \frac{\ell n(v_o/v_H)}{(v_o/v_H) - 1} \quad ; \tag{23}$$

this ratio is illustrated in Figure 1. Note that when $v_H/v_o = 1/2$, the displacement with drag is 69% of the displacement without drag, i.e., $s_H/s_{H_{C=0}} = 69\%$; when $v_H/v_o = 1/10$, then $s_H/s_{H_{C=0}} = 25.5\%$. Lastly, it is important to note that v_H/v_o can have any value from unity to zero: from Equation (16a), as $v_H/v_o \to 0$, we have $t_H \to \infty$; and, from Equation (18), as $v_H/v_o \to 0$, we have $s_H \to \infty$. Practically, of course this is meaningless, since we must use the time, t_H, in a restricted interval (since we neglect the effect of gravity).

In closure of this discussion on horizontal motion with drag, notice that we can find the altitude h_o required to have sufficient air resistance to slow the particle down to a given fraction v_H/v_o of its initial speed in a given specified time t_H; this is done by combining the two equations for C, Equations (1) and (16b), namely,

$$C = \frac{1}{2}\,(C_D A/m)\ \rho_o\ \exp(-h_o/H_o) = \frac{1}{v_o t_H}\left(\frac{v_o}{v_H} - 1\right) \tag{24}$$

and solving for the density function,

$$\rho(h_o) = \rho_o\ \exp(-h_o/H_o) \tag{25a}$$

in terms of v_o, t_H, and v_o/v_H in the form

$$\rho(h_o) = \frac{2}{v_o t_H}\left(\frac{v_o}{v_H} - 1\right)(m/C_D A) \quad . \tag{25b}$$

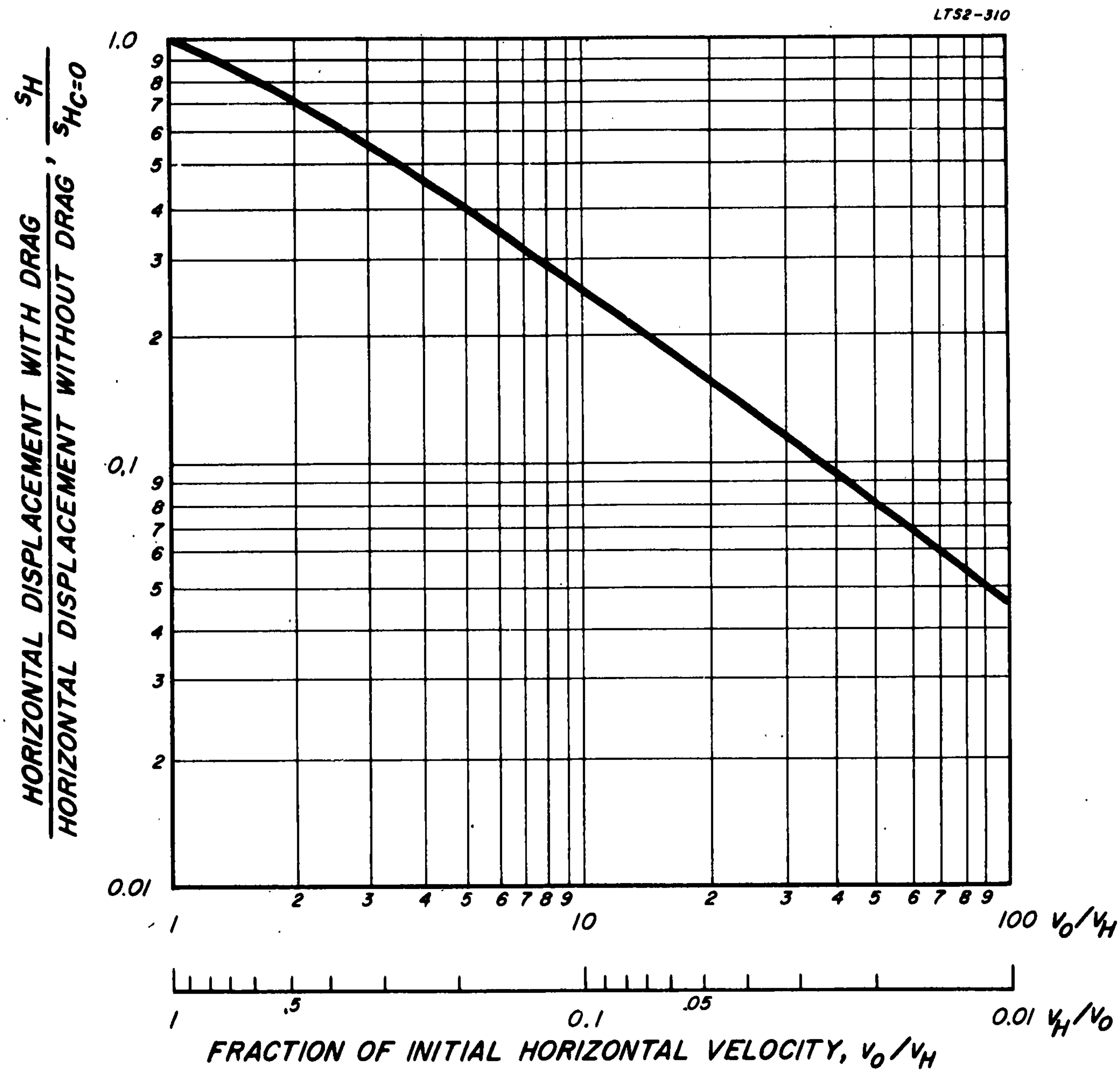

Figure 1. Variation of Horizontal Displacement with Horizontal Velocity, with Drag only on Particle.

The right hand side of Equation (25b) is known as soon as t_H and v_o/v_H are selected for a particular release v_o and particle, m/C_DA; hence $\rho(h_o)$ is known. To find h_o one can use Equation (25a) for specified values of ρ_o and H_o, which can be found in a standard model atmosphere handbook. One can use the ARDC Model Atmosphere tables directly, entering with $\rho = \rho(h_o)$ and reading out h_o, the altitude corresponding. In the Appendix will be found results of calculations for particles of radii 2μ, .5μ and .03μ, for $C_D = 1$ and $C_D = 2$, for a range of initial velocities v_o, from $v_o = 10^2$ cm/sec to $v_o = 5 \times 10^5$ cm/sec.

Equation (25b) defines a formula for the atmospheric density, $\rho(h_o)$, required to slow a particle down from an initial speed v_o to the value v_H, in the time t_H. Equation (25a) can be used to determine the corresponding altitude h_o (when the ARDC Atmosphere Model tabulation is not available).

II. UPWARD MOTION WITH DRAG

In this section we consider Case II, motion in an upward direction with drag only. From Equation (11), we assert that v_U, the velocity upwards, is defined by the differential equation

$$dv_U/dt = -C \exp(-s_U/H_o)\, v_U^2, \qquad v_U = ds_U/dt, \qquad s_U = h - h_o \qquad (26a)$$

with the initial conditions

$$v_U = v_o, \qquad s_U = 0, \qquad \text{when } t = 0 \quad . \qquad (26b)$$

Since

$$dv_U/dt = (dv_U/ds_U)\,(ds_U/dt) = v_U\,(dv_U/ds_U) \qquad (27)$$

we can express Equation (26a) in the alternate form,

$$(1/v_U)\,(dv_U/ds_U) = -C \exp(-s_U/H_o) \qquad (28)$$

which has the integral

$$\ell n(v_U/v_o) = -C\,H_o[1 - \exp(-s_U/H_o)] \quad , \qquad (29)$$

Solving for the displacement s_U/H_o, in Equation (29), we see that

$$\exp(-s_U/H_o) = 1 - \frac{1}{CH_o}\,\ell n(v_o/v_U) \quad , \qquad (30)$$

Since the left hand side of Equation (30) is never negative, it is clear that v_o/v_U is restricted to the range

$$1 \leqq (v_o/v_U) \leqq \exp(CH_o) \tag{31}$$

for all distances $s_U \geqq 0$. In other words, the minimum value that v_U/v_o can attain is $\exp(-CH_o)$, that is,

$$(v_U/v_o)_{min} = \exp(-CH_o) \tag{32}$$

When $v_U/v_o \to \exp(-CH_o)$ we see from Equation (30) that $\exp(-s_U/H_o) \to 0$, which means that $s_U/H_o \to \infty$. As the particle moves upwards it is slowed down to the asymptotic $v_U = v_o \exp(-CH_o)$ as it recedes away from the initial point.

Equation (30) defines a formula for the vertical accent of a particle, s_U, in terms of the speed ratio $v_o/v_U \leqq \exp(CH_o)$.

It is an easy matter to compare the displacements in the horizontal, s_H/H_o, and vertical, s_U/H_o, directions under the condition that the speeds are equal,

$$v_H/v_o = v_U/v_o \leqq \exp(CH_o) \tag{33}$$

To this end we refer to Equation (19) for v_H/v_o,

$$v_H/v_o = \exp(-Cs_H) = v_U/v_o, \qquad \text{from Equation (33)}$$

and substitute $v_o/v_U = \exp(Cs_H)$ in Equation (30); the resulting equation is

$$\exp(-s_U/H_o) = 1 - s_H/H_o, \qquad 0 \leqq s_H \leqq H_o \quad . \tag{34}$$

Equation (34) is a formula relating s_H/H_o and s_U/H_o for the condition that $v_H/v_o = v_U/v_o$.

To find the time required for the particle to slow down from v_o to v_U, in ascending the distance s_U above h_o, we turn to Equation (29) and write it in the form

$$ds_U/dt = v_o \exp\left(-[1 - \exp(-s_U/H_o)]\ CH_o\right) \tag{35}$$

which can be solved for the differential of time, dt, in the form

$$dt = \frac{1}{v_o} \exp\left([1 - \exp(-s_U/H_o)]\ CH_o\right) ds_U \tag{36}$$

Integration yields the formula

$$t_U = \frac{1}{v_o} \exp(CH_o) \int_o^{s_U} \exp[-CH_o \exp(-z/H_o)]\ dz \tag{37}$$

Equation (37) is a formula for the time $t = t_U$ required for the particle to ascend a distance s_U.

To find t_U in terms of v_U/v_o, we again turn to Equation (24) and let

$$\zeta = [1 - \exp(-s_U/H_o)]\ CH_o = \ell n(v_o/v_U) \tag{38}$$

so that

$$d\zeta/dt = C\ \exp(-s_U/H_o)\ \frac{ds_U}{dt} = C[1 - (\zeta/CH_o)]\ \frac{ds_U}{dt}$$

hence

$$ds_U/dt = \frac{1}{C[1 - (\zeta/CH_o)]}\ \frac{d\zeta}{dt} = v_o\ \exp(-\zeta), \text{ from Equation (35)}, \tag{39}$$

We rewrite Equation (39) as

$$dt = \frac{H_o}{v_o}\ \frac{\exp(\zeta)}{CH_o - \zeta}\ d\zeta \tag{40}$$

where $\zeta = \ell n(v_o/v_U)$ so that $\zeta = 0$ when $t = 0$ and $v_U = v_o$.

By integration of Equation (40) we obtain the formula

$$t_U = \frac{H_o}{v_o} \int_o^{\ell n(v_o/v_U)} \frac{\exp(x)}{CH_o - x}\ dx, \qquad \ell n(v_o/v_U) \leqq CH_o \quad . \tag{41}$$

<u>Equation (41) is a formula for the time $t = t_U$ required for the particle to slow down from v_o to v_U, with $v_U/v_o = \exp(-CH_o)$.</u>

We now take up a brief study of this formula, Equation (41), for t_U. For one thing, we can show that $t_U \to \infty$ as $v_o/v_U \to \exp(CH_o)$: from Equation (41) (see Figure 2)

$$\lim_{(v_o/v_U) \to \exp(CH_o)} t_U = \lim_{\epsilon \to 0} \frac{H_o}{v_o} \int_o^{CH_o-\epsilon} \frac{\exp(x)}{CH_o-x}\, dx$$

$$= \lim_{\epsilon \to 0} \frac{H_o}{v_o} \int_o^{CH_o-\epsilon} \frac{dx}{CH_o-x} = \frac{H_o}{v_o} \lim_{\epsilon \to 0} \left|\ell n \frac{\epsilon}{CH_o}\right| = \infty \qquad (42)$$

<u>That is to say, it takes an infinite time for the particle to reach its minimum velocity, $v_o \exp(-CH_o)$ as would be expected intuitively.</u>

In order to evaluate t_U for $v_U/v_o \leqq \exp(-CH_o)$ we set

$$CH_o - x = y \qquad (43)$$

in Equation (44) and make the appropriate reductions; we find that

$$t_U = \frac{H_o}{v_o} \exp(CH_o) \int_{CH_o-\beta}^{CH_o} \frac{\exp(-y)}{y}\, dy \qquad (44)$$

where

$$\beta = \ell n(v_o/v_U), \qquad v_U = v_o \exp(-\beta), \qquad 0 \leqq \beta < CH_o \quad , \qquad (45)$$

If we now let

$$\beta = (1-\mu)\ CH_o, \qquad 0 \leqq \mu \leqq 1 \qquad (46)$$

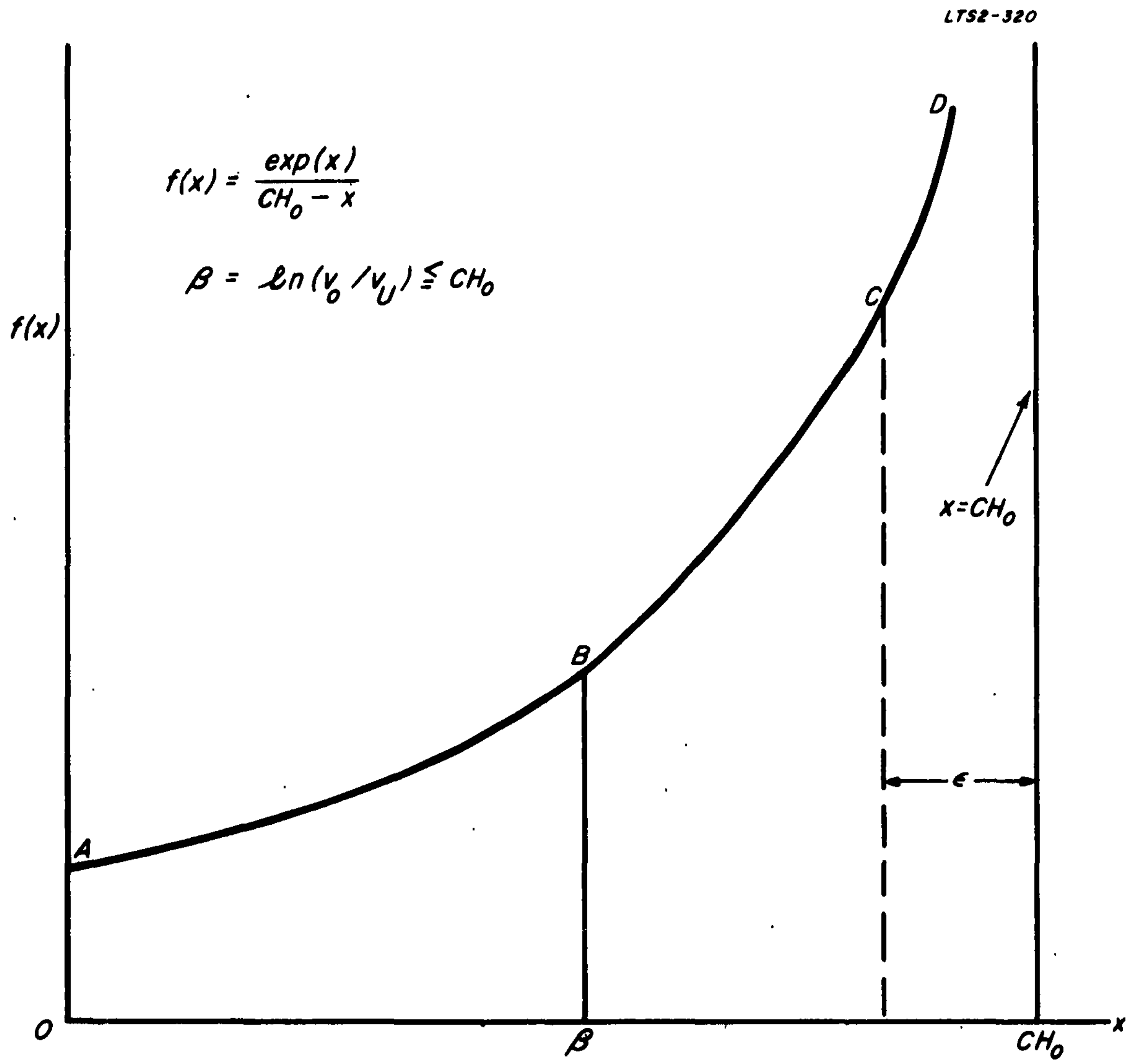
LTS2-320
$f(x) = \frac{\exp(x)}{CH_o - x}$
$\beta = \ln(v_o/v_U) \leq CH_o$
f(x)
D
C
B
A
$x = CH_o$
ϵ
O
β
CH_o
x

Figure 2.

so that

$$v_o/v_U = \exp[(1-\mu)\ CH_o] \tag{47}$$

then Formula (44) takes the alternate form

$$t_U = \frac{H_o}{v_o} \exp(CH_o) \int_{\mu CH_o}^{CH_o} \frac{\exp(-y)dy}{y} \quad . \tag{48}$$

In the handbook by Jahnke and Ende, "Tables of Fundations", page 1, there is defined the function

$$-Ei(-x) = \int_x^{\infty} \frac{\exp(-y)dy}{y} \tag{49}$$

which can be used to evaluate t_U. For note that since

$$\int_{x_1}^{x_2} g(x)\ dx = \int_{x_1}^{\infty} g(x)\ dx - \int_{x_2}^{\infty} g(x)\ dx$$

we can write Equation (48) in the form

$$t_U = \frac{H_o}{v_o} \exp(CH_o) \left([-Ei(-\mu CH_o)] - [-Ei(-CH_o)] \right) \tag{50}$$

Jahnke and Ende give only a small graph of the function -Ei(-x); reading from this graph one can construct the following table of values (Table 1)

TABLE 1

VALUES OF -Ei(-x)

$$-Ei(-x) = \int_{o}^{\infty} \exp(-y)\ y^{-1}\ dy$$

x	-Ei(-x)	x	-Ei(-x)	x	-Ei(-x)
0	∞	.6	.45	1.2	.15
.1	1.70	.7	.40	1.3	.12
.2	1.20	.8	.27	1.4	.11
.3	.90	.9	.24	1.5	.10
.4	.70	1.0	.20	1.6	.09
.5	.55	1.1	.16		

On the other hand, the formula for t_U in Equation (41) can be approximated analytically when

$$\ell n(v_o/v_U) \ll CH_o \tag{51}$$

for then

$$t_U = \frac{1}{Cv_o} \int_{o}^{\ell n(v_o/v_U)} \exp(x) \quad (1 + \frac{x}{CH_o} + \ldots) \tag{52}$$

so that

$$\left.\begin{aligned} t_U &= \frac{1}{Cv_o} \left(\exp(x) + [\exp(x)/CH_o]\ (x-1)\right)_{x=0}^{x=\ell n(v_o/v_U)}, \\ v_o/v_U &\ll \exp(CH_o) \qquad . \end{aligned}\right\} \tag{53}$$

Simplification of Equation (53) results in the equation

$$t_U = \frac{1}{Cv_o} \; [(v_o/v_U)-1][1-(1/CH_o)] + \frac{1}{Cv_o}\frac{1}{CH_o} \; \ell n(v_o/v_U) \tag{54}$$

or

$$t_U = \frac{1}{Cv_o}\left[(v_o/v_U)-1\right] + \frac{1}{Cv_o}\frac{1}{CH_o}\left(\left[(v_o/v_U)\;\ell n(v_o/v_U)\right] - \left[(v_o/v_U)+1\right]\right) \tag{55}$$

<u>Equation (54) is a formula for the time t_U required for the ascending particle to slow down from v_o to v_U, $v_o/v_U \ll \exp(CH_o)$.</u>

If we now compare t_U with t_H by requiring that $v_o/v_H = v_o/v_U$, it is seen that Equation (54) is equivalent to

$$\left.\begin{aligned} t_U &= t_H + \frac{t_H}{CH_o}\left[\frac{1}{1-(v_H/v_o)}\right][\ell n(v_o/v_H)-1+(v_H/v_o)], \\ v_o/v_H &= v_o/v_U \leqq \exp(CH_o) \end{aligned}\right\} \tag{56}$$

It is clear that if $v_o/v_H \geqq e$ then $t_U > t_H$. It can be shown, however, that $t_U > t_H$ even when $1 \leqq v_o/v_H \leqq e$; for let $v_o/v_H = \exp[1-\mu(v_H/v_o)]$, $0 < \mu < 1$, so that $\ell n(v_o/v_H) = 1-\mu(v_H/v_o)$ then in Equation (56)

$$t_U = t_H + \frac{t_H}{CH_o}\left[\frac{v_H/v_o}{1-(v_H/v_o)}\right](1-\mu) > t_H \tag{57}$$

In other words, for the same drop in speed it takes a longer time in ascending than in moving horizontally, $t_U > t_H$ when $v_o/v_H = v_o/v_U \lessapprox \exp(CH_o)$. This is of course to be expected intuitively and serves as a check on our solution.

Equation (54) can be written in the form

$$\left.\begin{aligned} t_U &= [1-(1/CH_o)]\, t_H + (1/CH_o)(v_o/v_H)(s_H/H_o), \\ (v_o/v_H) &= (v_o/v_U) \ll \exp(CH_o) \end{aligned}\right\} \tag{58}$$

In order to clarify the previous discussion and to illustrate the use of the formulae, we consider

Example 1

We take a particle of radius $.5\mu = 5 \times 10^{-5}$ cm, with mass $m = 1.84 \times 10^{-12}$ cm and resisting area $A = 7.85 \times 10^{-9}$ cm^2 so that

$$(m/C_D A) = 1.2 \times 10^{-4} \text{ gm/cm}^2 \qquad \text{with } C_D = 2 \tag{59}$$

We stipulate that

$$(v_o/v_H0 = (v_o/v_U) = 10, \qquad \text{with } v_o = 2.5 \times 10^3 \text{ cm/sec} \tag{60}$$

so that $v_H = v_U = 250$ cm/sec; we also set $t_H = 10$ sec. The required density is

$$\rho(h_o) = \frac{2}{v_o t_H}\left(\frac{v_o}{v_H} - 1\right)\left(\frac{m}{C_D A}\right) = \frac{2}{2.5 \times 10^4} \times 9 \times (1.2 \times 10^{-4})$$

$$= 8.64 \times 10^{-8} \text{ gm/cm}^3 \tag{61}$$

The ARDC Model Atmosphere tables give $h_o = 71$ km $= 7.1 \times 10^6$ cm as the corresponding attitude. The constant C has the value

$$C = \frac{1}{2}(C_D A/m)\,\rho(h_o) = \frac{1}{2}\,\frac{8.64 \times 10^{-8}}{1.2 \times 10^{-4}} = 3.6 \times 10^{-4}/\text{cm} \quad . \tag{62}$$

To find the scale height H_o at $h_o = 71$ km we look up in the ARDC manual and select two adjacent entries: $h_1 = 71$ km, $\rho_1 = 8.64 \times 10^{-8}$ gm/cm^3, $h_2 = 75.5$ km, $\rho_2 = 4.32 \times 10^{-8}$ gm/cm^3; we then use $\rho(h) = \rho_o \exp(-h/H_o)$ for each of these and form the quotient

$$(\rho_1/\rho_2) = \exp[(h_2-h_1)/H_o] \quad ; \tag{63}$$

hence, with $(\rho_1/\rho_2) = 2$, $(h_2-h_1) = 4.5$ km, we have

$$\exp(4.5/H_o) = 2 = \exp(.693); \qquad H_o = 6.49 \text{ km.} \tag{64}$$

With $H_o = 6.49$ km, we have

$$CH_o = (3.6 \times 10^{-4})(6.49 \times 10^5) = 233.64 \quad . \tag{65}$$

Since $\ell n(v_o/v_U) = \ell n(10) = 2.303 \ll 233.64 = CH_o$, we have $(v_o/v_U) \ll \exp(CH_o)$ so that it is legitimate to use approximate formulas. We need, however, the horizontal distance s_H; we use

$$s_H = \left[\frac{\ell n(v_o/v_H)}{(v_o/v_H)-1}\right] v_o t_H = \left(\frac{2.302}{9}\right)(2.5 \times 10^4) = 6394 \text{ cm} \tag{66}$$

from Equation (22). Hence in Equation (58)

$$t_U = \left(1 - \frac{1}{CH_o}\right) t_H + \frac{1}{CH_o} \frac{v_o}{v_H} \frac{s_H}{H_o} = (1-.00428)10 + \left(\frac{1}{233.64}\right)$$

$$\times (10) \left(\frac{6.394 \times 10^3}{2.5 \times 10^3}\right)$$

which works out to be

$$t_U = 9.9572 + .10947 = 10.06667 \text{ sec} \tag{67}$$

Further, from Equation (34), we have

$$\exp(-s_U/H_o) = 1 - (s_H/H_o) = 1 - .009846 = .99016 \tag{68}$$

so that $s_U/H_o = .009887$ and $s_U = 6417.75$ cm

In summary: if $v_o = 2.5 \times 10^3$ cm/sec, $v_o/v_H = v_o/v_U = 10$ in $t_H = 10$ sec, and if $m/C_DA = 1.24 \times 10^{-4}$ gm/cm^2, $C = 3.6 \times 10^{-4}$/cm, $H_o = 6.49 \times 10^5$ cm, then $\rho(h_o) = 8.64 \times 10^{-8}$ gm/cm^3, $h_o = 7.1 \times 10^6$ cm, $s_H = 6.394 \times 10^3$ cm, $s_U = 6.418 \times 10^3$ cm, $t_U = 10.0667$ sec.

Example 2 (Gegenbeispiel)

We now take up an illustration in which our formulas fail: with the same partice as in Example 1 we now let $v_o = 5 \times 10^5$ cm/sec and put the same velocity drop ratio requirement as before, $v_H/v_o = v_U/v_o = 1/10$ in $t_H = 10$ sec. We find that $\rho(h_o) = 4.32 \times 10^{-10}$ gm/cm^3 and $h_o = 99$ km. The constant C is now 1.8×10^{-6}/cm; the scale height H_o is now 4.48×10^5 cm, and $CH_o = .806$. Since $\ell n(v_o/v_U) = \ell n(10) = 2.302$, we see that $(v_o/v_U) > \exp(CH_o)$ in violation of the minimum velocity drop ratio restriction. That is to say, at the height $h_o = 99$ km it is impossible for the particle, beginning with an upward velocity $v_o = 5 \times 10^5$ cm/sec to slow down to a speed $v_U = (1/10)v_o = 5 \times 10^4$ cm/sec: the minimum value of (v_U/v_o) attainable is $\exp(-CH_o) = \exp(-.806) = .4466$. At such a high altitude ($h_o = 99$ km) and with such an initial speed ($v_o = 5 \times 10^5$ cm/sec), the drag resistance can only slow it down to the value $v_U = .4466\ v_o$.

If gravity acts on the particle it will halt it in time (if v_o is not an escape velocity). In the next section we bring the gravity into play, in combined action with the drag, for particles released in an upward direction.

III. UPWARD MOTION WITH DRAG AND GRAVITY

In this section we consider Case IV, motion of a particle with an initial upward velocity, with drag and gravity combined, acting to slow it down. The upward velocity is now designated by

$$v_V = (ds_V/dt), \qquad s_V = h - h_o \quad , \tag{69}$$

The dynamic equation of motion is expressed in Equation (13) as

$$(dv_V/dt) = - C \exp(-s_V/H_o)\, v_V^2 - g \tag{70a}$$

with the initial conditions

$$v_V = v_o, \qquad s_V = 0, \qquad \text{when } t = 0 \quad . \tag{70b}$$

To integrate Equation (70a), we begin by replacing dv_V/dt by $v_V dv_V/ds_V$, and multiplying through by ds_V; we find that

$$v_V\, dv_V = - C\, v_V^2 \exp(-s_V/H_o)\, ds_V - g\, ds_V \tag{71}$$

or

$$d(g\, s_V + \frac{1}{2} v_V^2) = CH_o\, v_V^2\, d[\exp(-s_V/H_o)]$$

which can be written as

$$d(g\, s_V + \frac{1}{2} v_V^2) = 2CH_o(g\, s_V + \frac{1}{2} v_V^2)\, d[\exp(-s_V/H_o)]$$

$$- 2CH_o\, g\, s_V\, d[\exp(-s_V/H_o)] \tag{72}$$

In order to simplify the appearance of Equation (72) we let

$$u = g\ s_V + \frac{1}{2} v_V^{\ 2} \tag{73a}$$

$$w = \exp(-s_V/H_o), \qquad dw = -\ (w/H_o)\ d\ s_V \tag{73b}$$

we now have Equation (72) in the form

$$du = 2CH_o\ u\ dw - 2CH_o\ g\ s_V\ dw \tag{74}$$

The relationship $dw = -\ w\ d\left(s_V/H_o\right)$ reduces this to

$$du = -\ 2C\ u\ w\ d\ s_V + 2C\ g\ w\ s_V\ ds_V$$

which can be written in the standard form of a linear first order ordinary differential equation:

$$(du/ds_V) + 2C\ w\ u = 2C\ g\ w\ s_V \tag{75}$$

An integrating factor of Equation (75) is

$$u = \exp\left(\int 2C\ w\ ds_V\right) = \exp(-2CH_o w) \tag{76}$$

so that

$$\frac{d}{ds_V}\left[u\ \exp(-2CH_o w)\right] = 2C\ g\ w\ s_V\ \exp(-?CH_o w) \tag{77}$$

is an equivalent canonical form of Equation (75). Note that $w = 1$ and $u = (1/2)v_o^{\ 2}$ when $s_V = 0$, $(t = 0)$, so that by integration of Equation (77) we obtain an integral of motion, namely,

$$u \exp(-2CH_o w) - \frac{1}{2} v_o^2 \exp(-2CH_o)$$

$$= 2C\, g \int_o^{s_V} w\, s_V \exp(-2CH_o w)\, d\, s_V \quad . \tag{78}$$

Note that

$$\int_o^{s_V} w\, s_V \exp(-2CH_o w)\, d\, s_V = - H_o \int_1^{w} s_V \exp(-2CH_o w)\, dw$$

$$= \frac{1}{2C} s_V \exp(-2CH_o w) + \frac{H_o}{2C} \int_1^{w} \exp(-2CH_o x) \frac{dx}{x} \tag{79}$$

so that Equation (78) reduces to

$$u \exp(-2CH_o w) = \frac{1}{2} v_o^2 \exp(-2CH_o) + g\, s_V \exp(-2CH_o w)$$

$$+ g\, H_o \int_1^{w} \exp(-2CH_o x) \frac{dx}{x} \tag{80}$$

or, finally,

$$u = \frac{1}{2} v_o^2 \exp\left[-2CH_o(1-w)\right] + g\, s_V$$

$$+ g\, H_o \exp(2CH_o w) \int_1^{w} \exp(-2CH_o x) \frac{dx}{x} \tag{81}$$

But $u = g\ s_V + \frac{1}{2} v_V^2$, so that Equation (81) implies that

$$v_V^2 = v_o^2 \exp\left[-2CH_o(1-w_V)\right] - w\ g\ H_o \exp(2CH_o w_V)\int_{w_V}^{1} \exp(-2CH_o x)\frac{dx}{x} \quad (82a)$$

where

$$w = w_V = \exp(-s_V/H_o), \qquad (dw_V/dt) = -\frac{w_V}{H_o} v_V \quad (82b)$$

Equation (82) defines a formula for the velocity of ascent under the combined action of drag and gravity in terms of the ascent distance s_V.

As a check on this formula, it can be easily shown that it does satisfy the dynamic equation of motion Equation (70a).

With gravity and drag acting on the particle, in its ascent, it is possible that the particle will slow down to a halt, at which point $v_V = 0$, momentarily, before the particle then begins to descend, under the action of gravity. We designate this $v_V = 0$ condition by a sub asterisk on w_V, that is $w = w_*$ when $v_V = 0$. From Equation (82a), with $v_V = 0$, we see that w_* is defined by the integral equation

$$\int_{w_*}^{1} \exp(-2CH_o x)\ \frac{dx}{x} = \frac{v_o^2}{2g\ H_o} \exp(-2CH_o), \qquad w_* = \exp(-s_*/H_o) \quad (83)$$

Our problem to find w_* in terms of $v_o^2/2gH_o$ and $2\ CH_o$. In Equation (83) if we set $x = 1-y$, our definition of w_* becomes

$$\int_o^{1-w_*} \exp(2CH_o y)\,\frac{dy}{1-y} = \frac{v_o^2}{2gH_o} \tag{84}$$

Note that

$$1 - w_* = 1 - \exp(-s_*/H_o) = s_*/H_o + \ldots,\ \text{if } (s_*/H_o) \ll 1; \tag{85}$$

so that y is defined between D and $(s_*/H_o) \ll 1$, i.e., if condition Equation (85) holds, a binomial expression is valid on $(1-y)^{-1}$ in Equation (84). Indeed, with $(s_*/H_o) \ll 1$, Equation (84) becomes

$$\frac{v_o^2}{2gH_o} = \int_o^{s_*/H_o + \ldots} \exp(2CH_o y)\ (1 + y + \ldots)\ dy$$

$$= \left. \frac{1}{2CH_o} \exp(2CH_o y) + \frac{1}{(2CH_o)^2} \exp(2CH_o y)\quad (2CH_o y - 1) \right|_o^{y=s_*/H_o} . \tag{86}$$

With

$$\alpha = (1/2CH_o), \qquad \xi_* = 2Cs_*, \qquad \alpha\xi_* = (s_*/H_o) \ll 1, \tag{87}$$

Equation (86) can be written as

$$\frac{v_o^2}{2gH_o} = \alpha\ \exp(\xi_*)-1\ \ + \alpha^2\ \xi_*\ \exp(\xi_*) - \alpha^2\ \exp(\xi_*) + \alpha^2, \tag{88}$$

$$\alpha\xi_* = \ll 1$$

The inequality $\alpha\xi_* \ll 1$ permits a reduction of Equation (88): multiply Equation (88) through by ξ_* and rearrange terms, in the manner

$$\xi_* \frac{v_o^2}{2gH_o} = \left[\alpha\xi_* + (\alpha\xi_*)^2\right] \exp(\xi_*) - \alpha\xi_* - \alpha^2 \xi_* \exp(\xi_*) + \alpha^2 \xi_* \quad (89)$$

and since $(\alpha\xi_*)^2 \ll \alpha\xi_*$ we infer that it is valid to write Equation (89) as

$$\frac{v_o^2}{2gH_o} = \alpha \exp(\xi_*) - \alpha - \alpha^2 \exp(\xi_*) + \alpha^2$$

$$= \alpha\left[\exp(\xi_*)-1\right] - \alpha^2\left[\exp(\xi_*)-1\right]$$

or as

$$\frac{v_o^2}{2gH_o} = \alpha(1-\alpha)\left[\exp(\xi_*)-1\right], \qquad \alpha\xi_* \ll 1 \quad . \quad (90)$$

By simply rearrangement of terms in Equation (90) we obtain a formula for $\exp(\xi_*)$,

$$\exp(\xi_*) = 1 + \frac{1}{\alpha(1-\alpha)} \frac{v_o^2}{2gH_o} \quad (91)$$

and since $\xi_* = 2Cs_*$, $\alpha = (1/2)CH_o$, we infer that

$$\exp(2Cs_*) = 1 + \frac{(2CH_o)^2}{2CH_o-1} \frac{v_o^2}{2gH_o}, \qquad (s_*/H_o) \ll 1 \quad . \quad (92)$$

<u>Equation (92) defines a formula for the maximum height, $s_* = s_y$, attained by the particle in its ascent, under the combined action of gravity and drag, provided $(s_*/H_o) \ll 1$.</u>

We pause at this point to consider

Example 3

We take the data of Example 1: $v_o = 2.5 \times 10^3$ cm/sec, $H_o = 6.49 \times 10^5$ cm, $C = 3.6 \times 10^{-4}$/cm, $CH_o = 233.6$; then with $g = 980.616$ cm/sec^2,

$$\frac{v_o^2}{2gH_o} = .00491, \qquad \frac{(2CH_o)^2}{2CH_o - 1}\,\frac{v_o^2}{2gH_o} = 2.29924 \quad ; \tag{93}$$

hence in Equation (92)

$$\exp(2Cs_*) = 1 + 2.29924 = 3.2992 = \exp(1.194)$$

so that

$$(s_*/H_o) = (1.194/2CH_o) = (1.194/467.2) = .002555 \ll 1 \quad . \tag{94}$$

The criterion $(s_*/H_o) \ll 1$ is amply satisfied so that it is legitimate to use the formula for s_* in Equation (94). Hence, the maximum ascent distance which the particle can attain is

$$s_* = .002555\ H_o = 1658.19 \text{ cm} \tag{95}$$

when the initial upward velocity is $v_o = 2.5 \times 10^3$ cm/sec and the altitude is $h_o = 71$ km, under the combined action of drag and gravity. If gravity alone were to act on the particle, it would be stopped in $t_g = v_o/g = 2500/980.616 = 2.549$ sec and it would ascend a distance

$s_g = (1/2)v_o t_g = 3186.25$ cm. Hence the effect of drag combined with the gravity, is to cut the distance from $s_g = 3186.25$ cm to $s_* = 1658.19$ cm, or to reduce the free ascent by 1528.06 cm. Recall from Example 1 that the particle ascends a distance $s_U = 6418$ cm in $t_U = 10.0667$ sec while reducing the velocity from $v_o = 2500$ cm/sec to $v_U = 250$ cm/sec, under the action of drag alone.

Example 4 (Gegenbeispiel)

We take the data of Example 2: $v_o = 5 \times 10^5$ cm/sec, $C = 1.8 \times 10^{-6}$/cm, $h_o = 99$ km, $H_o = 4.48$ km, $CH_o = .806$; then

$$\frac{v_o^2}{2gH_o} = 284.543, \qquad \frac{(2CH_o)^2}{2CH_o - 1}\ \frac{v_o^2}{2gH_o} = 1208.163$$

hence

$$\exp(2Cs_*) = 1 + 1208.163 = 1209.163 = \exp(7.0977)$$

so that

$$(s_*/H_o) = (7.0977/1.612) = 4.403 > 1 \quad ; \qquad (96)$$

this value of $(s_*/H_o) = 4.403$ is a violation of the restriction that $(s_*/H_o) \ll 1$: Equation (92) cannot be used in Example 4.

In working Example 3 we did not consider the time required to reach the maximum ascent. This points up the need to determine the time t_V required for the particle to slow down from v_o to v_V while ascending a distance s_V under the combined action of aerodynamic drag and gravity. To find t_V we need dt_V which we can define by use of Equation (82):

$$dt_V = - \frac{H_o}{w_V v_V} dw_V, \qquad w = \exp(-s_V/H_o) \qquad . \tag{97}$$

At least theoretically, we have t_V defined by the integral of Equation (97), namely,

$$t_V = H_o \int_{w_V}^{1} \frac{dw_V}{w_V v_V} , \qquad w_V = \exp(s_V/H_o) \qquad , \tag{98}$$

where v_V is defined by Equation (82a).

<u>Equation (98) defines a formula for the particle to ascend a distance s_V against the combined actions of aerodynamic drag and gravity.</u>

On the other hand since $v_V = ds_V/dt$, we can write $dt = ds_V/v_V$ and then express t_V in the form

$$t_V = \int_{o}^{s_V} \frac{ds_V}{v_V} \qquad . \tag{99}$$

In either case, by using Equations (98) or (99), it is necessary to have v_V as a function of s_V or $w_V = \exp(s_V/H_o)$. We, therefore, turn our attention to the reduction of the integral in Equation (82a):

$$I = \int_{w_V}^{1} \exp(-2CH_o x)\,\frac{dx}{x} = \exp(-2CH_o)\int_o^{1-w_V} \exp(2CH_o y)\,\frac{dy}{1-y} \quad . \tag{100}$$

Note that

$$1 - w_V = 1 - \exp(-s_V/H_o) = s_V/H_o + \dots \quad , \text{ if } s_V/H_o \ll 1 \ ,$$

so that, in Equation (100)

$$I = \exp(-2CH_o)\int_o^{s_V/H_o + \dots} [\exp(2CH_o y)](1+y)\,dy$$

$$= \exp(-1/\alpha)\ \alpha(1-\alpha)[\exp(\xi_V)-1] \tag{101}$$

by analogy with the development for w_x, where

$$\alpha = \frac{1}{2CH_o}\ , \qquad \xi_V = 2Cs_V\ , \qquad \alpha\xi_V = (s_V/H_o) \ll 1; \tag{102}$$

note that

$$\begin{aligned} &2CH_o\, w_V = \frac{1}{\alpha} - \xi_V + \dots, \qquad 1 - w_V = \alpha\xi_V + \dots \\ &2CH_o(1-w_V) = \xi_V, \qquad 2CH_o w_V - \frac{1}{\alpha} = -\xi_V + \dots \quad . \end{aligned} \tag{103}$$

When these abreviations are used in Equation (82a) our formula for v_V becomes

$$v_V^2 = v_o^2 \exp(-\xi_V) - 2gH_o \exp(-\xi_V)\, \alpha(1-\alpha)[\exp(\xi_V)-1], \quad \frac{s_V}{H_o} \ll 1 \quad (104)$$

Set

$$\gamma^2 = \frac{2gH_o}{v_o^2} \alpha(1-\alpha) \,, \qquad \alpha = \frac{1}{2CH_o} \qquad (105)$$

then Equation (104) can be written as

$$(v_V/v_o)^2 = \exp(-\xi_V)\left[1-\gamma^2[\exp(\xi_V)-1]\right], \qquad (s_V/H_o) \ll 1 \quad (106)$$

Taking a square root of both sides of Equation (106) we see that

$$v_V = (ds_V/dt) = v_o \exp(-Cs_V)\sqrt{(1+\gamma^2) - \gamma^2 \exp(2Cs_V)} \qquad (107)$$

so that by simple algebra

$$dt = \frac{1}{\gamma v_o} \exp(Cs_V) \frac{1}{\sqrt{\frac{1+\gamma^2}{\gamma^2} - \exp(2Cs_V)}} ds_V \,, \quad (s_V/H_o) \ll 1$$

and by integration

$$t_V = \frac{1}{\gamma v_o} \int_0^{s_V} \frac{\exp(Cx)}{\sqrt{\frac{1+\gamma^2}{\gamma^2} - \exp(2Cx)}} dx, \quad (s_V/H_o) \ll 1 \quad . \qquad (108)$$

If we now set $u = \exp(Cx)$, $\frac{1}{C} du = \exp(Cx)dx$, then Equation (108) reduces to

$$t_V = \frac{1}{Cv_o\gamma} \int_1^{\exp(Cs_V)} \frac{du}{\sqrt{\frac{1+\gamma^2}{\gamma^2} - u^2}} , \qquad (s_V/H_o) \ll 1 \tag{109}$$

which integrates at once to yield the formula

$$t_V = \frac{1}{Cv_o\gamma} \left[\sin^{-1}\left(\frac{\gamma\exp(Cs_V)}{\sqrt{1+\gamma^2}}\right) - \sin^{-1}\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right] , \quad (s_V/H_o) \ll 1 \tag{110}$$

where

$$\gamma^2 = \frac{2gH_o}{v_o^2} (1-\alpha), \qquad \alpha = \frac{1}{2CH_o}$$

It should be noted that Equation (91) can be written as

$$\exp(2Cs_*) = 1 + (1/\gamma^2) = (\gamma^2+1)/\gamma^2$$

so that

$$\exp(Cs_*) = \frac{\sqrt{\gamma^2+1}}{\gamma} , \qquad \text{or} \qquad \frac{\gamma\exp(Cs_*)}{\sqrt{1+\gamma^2}} = 1 \qquad ; \tag{111}$$

the implication is that, if $t_V = t_*$ when $s_V = s_*$ and $v_V = 0$, then in Equation (110)

$$t_* = \frac{1}{Cv_o\gamma} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right] = \frac{\cot^{-1}\gamma}{Cv_o\gamma} , \quad (s_V/H_o) \ll 1. \tag{112}$$

Equation (110) defines a formula for the time, t_V, it takes for a particle to ascend a distance s_V with a velocity drop from v_o to v_V, provided $s_V/H_o \ll 1$, with drag and gravity opposing the motion of the particle. Equation (112) defines a formula for the least time required for the particle to reach a position where it comes to a halt momentarily, $v_V = 0$, at a distance s_* above the starting point.

Example 5

We take the data of Examples 1 and 3: $h_o = 71$ km, $v_o = 2.5 \times 10^3$ cm/sec, $C = 3.6 \times 10^{-4}$/cm, $H_o = 6.49 \times 10^5$ cm, $2CH_o = 467.2$, $s_V = s_* = 1658.19$ cm, $\exp(2Cs_*) = 3.2992$, $\exp(Cs_*) = 1.81638$, $(2gH_o/v_o^2) = 203.6528$, $\alpha = (1/2CH_o) = .0021404$, $1-\alpha = .99786$, $\alpha(1-\alpha) = .0021358$, $\gamma^2 = .4349616$, $\gamma = .65952$, $\cot^{-1}\gamma = .98756$, $\sqrt{1+\gamma^2} = 1.1979$, $\gamma/\sqrt{1+\gamma^2} = .550563 = \sin(.583044^{(r)})$; $Cv_o\gamma = .593568$, $(1/Cv_o\gamma) = 1.684726$; $(\gamma/\sqrt{1+\gamma^2})\exp(Cs_*) = (.550562)(1.81632) = 1 = \sin(1.57079^{(r)})$; hence

$$t_V = t_* = 1.684726\ (1.57079 - .583044) = 1.6641 \text{ sec } \frac{\cot^{-1}\gamma}{Cv_o\gamma} = \frac{.98756}{.59357} \ ,$$

We conclude that it takes $t_* = 1.6641$ sec for the particle to ascend to its maximum displacement $s_* = 1658.19$ cm, where the particle momentarily comes to rest $v_V = 0$, under the combined action of drag and gravity, beginning with a velocity $v_o = 2.5 \times 10^3$ cm/sec upwards. As a point of comparison, to ascend a distance $s_* = 1658.2$ cm with an initial velocity $v_o = 2500$ cm/sec and be acted upon by gravity alone, it takes .7836 sec.

We now let $v_o/v_V = 10$ and seek the distance s_V. From Equation (106) by simple algebra we derive the formula

$$\exp(Cs_V) = \sqrt{\frac{1+\gamma^2}{(v_V/v_o)^2 + \gamma^2}} \; , \qquad (s_V/H_o) \ll 1 \quad . \tag{113}$$

<u>Equation (113) defines a formula for the distance of ascent, s_V, of a particle under the combined action of drag and gravity, in terms of the velocity ration v_V/v_o, provided $s_V \ll H_o$.</u>

Thus with $1+\gamma^2 = 1.434916$, $(v_V/v_o)^2 + \gamma^2 = .01 + .4349616 = .444916$, we have $\exp(Cs_V) = \sqrt{3.225139} = 1.7958 = \exp(.58545)$ so that $(s_V/H_o) = (.58545/CH_o) = (.58545/233.6) = .0025062 < 1$, $s_V = 1626.52$ cm; that is to say: the particle ascends a distance $s_V = 1626.52$ cm against drag and gravity while dropping its speed from $v_o = 2500$ cm/sec to $v_V = 250$ cm/sec. To find the corresponding time, t_V, we use Equation (110), wherein $Cs_V = .58545$, $\exp(Cs_V) = 1.7958$, $\gamma \exp(Cs_V)/\sqrt{1+\gamma^2} = .988701 = \sin(81\ .379) = \sin(1.4203^{\,r})$, $(\gamma/\sqrt{1+\gamma^2}) = .550563 = \sin(.58304^{(r)})$, hence $t_V = 1.684726\ (1.4203 = .58304) = 1.4105$ sec; it takes $t_V = 1.4105$ sec for the particle to lose 90% of its initial speed, to go from $v_o = 2500$ cm/sec to $v_V = 250$ cm/sec, working against drag and gravity, in going a distance $s_V = 1626.52$ cm.

IV. DOWNWARD MOTION WITH DRAG

In this section we consider Case III, motion downwards with drag only. We designate the velocity vector downwards by v_D and define it by the differential Equation (12a),

$$dv_D/dt = - C \exp(s_D/H_o)\, v_D^2 , \quad v_D = ds_D/dt , \quad s_D = h_o - h \tag{114a}$$

and have it satisfy the initial conditions

$$v_D = v_o , \quad s_D = 0 , \quad \text{when } t = 0. \tag{114b}$$

Following the techniques used previously, we find that

$$(1/v_D)(dv_D/ds_D) = - C \exp(s_D/H_o) \tag{115}$$

is an equivalent form of Equation (114a) and that

$$\ell n(v_D/v_o) = - [\exp(s_D/H_o)-1]\, CH_o \tag{116a}$$

and

$$(v_D/v_o) = \exp\left\{-[\exp(s_D/H_o)-1]\, CH_o\right\} \tag{116b}$$

are first integrals.

The equations in Equation (116) defined a relationship between the velocity ratio v_D/v_o and the descent distance s_D for a particle descending into an exponential atmosphere under the action of drag alone.

By simple rearrangement of terms we transform Equation (116) into

$$\exp(s_D/H_o) = 1 + \frac{1}{CH_o}\,\ell n(v_o/v_D) \quad . \tag{117}$$

It is of basic interest to compare the values of the displacements s_H, s_U, and s_D at the instants their corresponding speeds are equal, that is when $v_o/v_H = v_o/v_U = v_o/v_D \leq \exp(CH_o)$.

From Equation (19), $\ell n(v_o/v_H) = Cs_H$, so that in Equation (117)

$$\exp(s_D/H_o) = 1 + \frac{s_H}{H_o} \quad ; \tag{118}$$

since $\exp(-s_U/H_o) = 1 - (s_H/H_o)$, from Equation (34), it follows that

$$\begin{gathered}\exp(s_D/H_o) = 2 - \exp(-s_U/H_o), \\ (v_o/v_H) = (v_o/v_D) = (v_o/v_U) \leqq \exp(CH_o) \quad .\end{gathered} \tag{119}$$

In Figure 3 is shown a comparison of the displacements s_U/H_o and s_D/H_o with s_H/H_o for the condition of equal speeds.

To find a time dependency relationship, we begin with Equation (116b),

$$v_D = (ds_D/dt) = v_o \exp\left\{-[\exp(s_D/H_o)-1]\ CH_o\right\} \tag{116b}$$

and express the element of time, dt, as

$$dt = \frac{1}{v_o} \exp\left\{[\exp(s_D/H_o)-1]\ CH_o\right\}\ ds_D \tag{120}$$

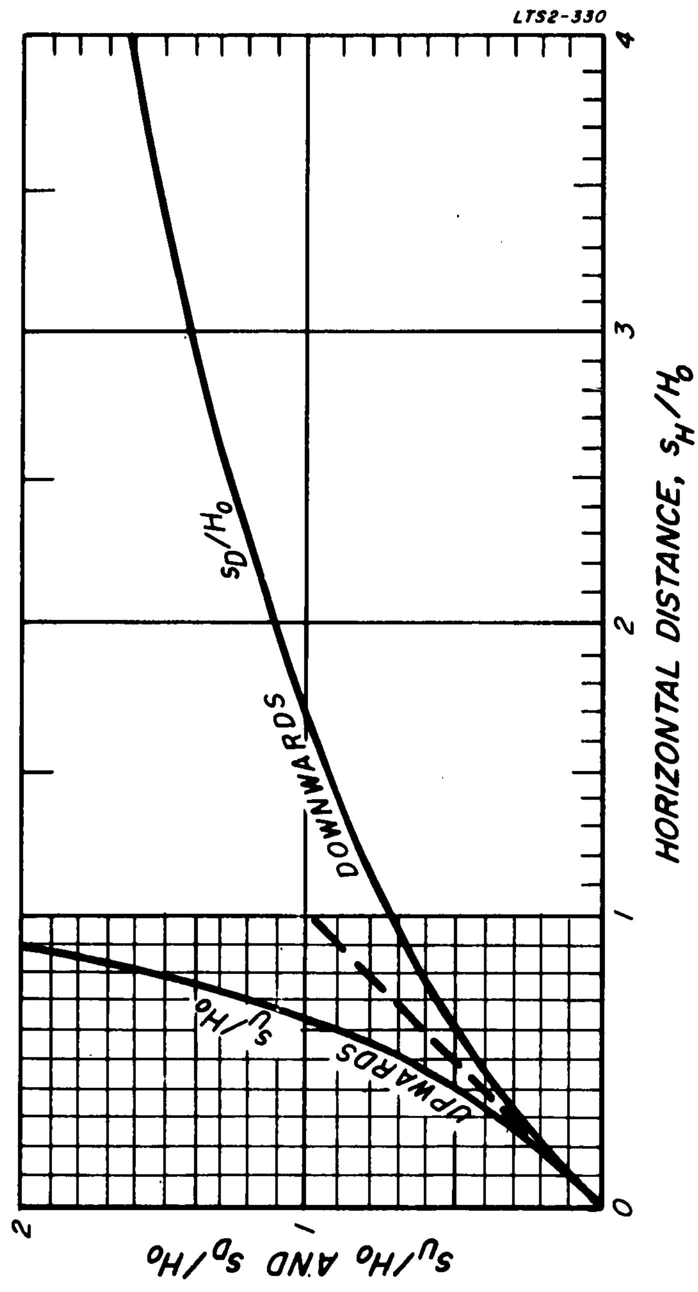

Figure 3. Comparison of Upward and Downward Distances s_U/H_o and s_D/H_o, with Drag only, with Horizontal Distance s_H/H_o, for Equal Velocity Ratios, $v_H/v_o = v_U/v_o = v_o/H_o$.

which can be integrated directly, yielding the formula

$$t_D = [\exp(-CH_o)/v_o] \int_o^{s_D} \exp[\exp(\zeta/H_o)\ CH_o]\ d\zeta \quad . \tag{121}$$

To find an alternate formula for t_D we let

$$\zeta = [\exp(s_D/H_o)-1]\ CH_o = \ell n(v_o/v_D) \tag{122}$$

in Equation (116b), so that

$$(d\zeta/dt) = C \exp(s_D/H_o) \frac{ds_D}{dt} = C\ [1 \div (\zeta/CH_o)] \frac{ds_D}{dt} \quad , \tag{123}$$

hence, by solving for ds_D/dt in Equation (123) and using the velocity relationship in Equation (116b), one finds that

$$(ds_D/dt) = \frac{1}{C} \frac{1}{1 + \zeta/CH_o} \frac{d\zeta}{dt} = v_o \exp(-\zeta) \tag{124}$$

which implies that

$$dt = \frac{H_o}{v_o} \frac{\exp(\zeta)}{CH_o \div \zeta} d\zeta \quad . \tag{125}$$

and that

$$t_D = \frac{H_o}{v_o} \int_o^{\ell n(v_o/v_D)} \frac{\exp(y)}{CH_o + y} dy \tag{126}$$

since $\zeta = 0$ when $t = 0$, from Equation (122).

Equations (121) and (126) are formulas for the time, t_D, of descent, against aerodynamic drag, in terms of the distance s_D, in the former, and in terms of the velocity ration drop v_D/v_o, in the latter.

In the event that $v_o/v_D \ll \exp(CH_o)$, the kernel of the integral in Equation (126) can be expanded in a series and integrated termwise, thusly,

$$t_D = \frac{1}{C}\left(\frac{1}{v_D} - \frac{1}{v_o}\right) - \frac{1}{Cv_o}\,\frac{1}{CH_o}\left(\frac{v_o}{v_D}\,\ell n\,\frac{v_o}{v_D} + 1 - \frac{v_o}{v_D}\right),\quad \frac{v_o}{v_D} \ll \exp(CH_o)\ . \qquad (127)$$

Equation (127) is a formula for the time, t_D, required for the particle to slow down from a downward velocity v_o to the value v_D, provided $v_o/v_D \ll \exp(CH_o)$. To appraise this formula suppose that $v_o/v_D = v_o/v_H$ then, from Equation (15), the first term on the right hand side of Equation (127) is

$$t_H = \frac{1}{C}\left(\frac{1}{v_H} - \frac{1}{v_o}\right)$$

so that

$$t_D = t_H - \frac{1}{Cv_o}\,\frac{1}{CH_o}\left(\frac{v_o}{v_D}\,\ell n\,\frac{v_o}{v_D} + 1 - \frac{v_o}{v_D}\right),\quad \frac{v_o}{v_D} = \frac{v_o}{v_H} \ll \exp(CH_o)\ , \qquad (128)$$

An alternate form of Equation (128) is

$$t_D = t_H - \frac{t_H}{CH_o\left(1 - \frac{v_H}{v_o}\right)}\left(\ell n\,\frac{v_o}{v_D} - 1 + \frac{v_D}{v_o}\right),\quad \frac{v_o}{v_D} = \frac{v_o}{v_H} \ll \exp(CH_o) \qquad (129)$$

Example 6

We take the data of Example 1: $h_o = 71$ km, $v_o = 2.5 \times 10^5$ cm/sec, $H_o = 6.49 \times 10^5$ cm, $C = 3.6 \times 10^{-4}$/cm, $CH_o = 233.6$; we choose $v_o/v_D = v_o/v_H = 1/10$, $\ell n(v_o/v_D) = 2.30259$, $s_H = 6.394 \times 10^3$ cm, $s_H/H_o = .009852$; hence

$$\exp(s_D/H_o) = 1 + (s_H/H_o) = 1.009852 \tag{130}$$

so that $s_D/H_o = .0098035$ and $s_D = 6362.47$ cm (which compares with $s_U = 6418$ cm and $s_V = 1628.6$ cm) is the distance the particle descends against drag alone, in reducing the downward velocity from $v_o = 2500$ cm/sec to $v_D = 250$ cm/sec. To find the time, t_D, we use Equation (127): with $v_o/v_D = v_o/v_H$ we have $t_H = 10$ sec (see Example 1), or else note that $Cv_o = .9$, $v_o/v_D - 1 = 9$; further $v_o/v_D \; \ell n(v_o/v_D) = 20.72331$, so that

$$t_D = 10 - .055761 = 9.9442 \text{ sec} \quad . \tag{131}$$

The time required for the particle to descend the distance $s_D = 6362.47$ cm while going from $v_o = 2500$ cm/sec to $v_D = 250$ cm/sec, is $t_D=9.9442$ sec (compare with $t_H = 10$ sec, $t_U = 10.0667$ sec for the condition $v_o/v_D = v_o/v_U = v_o/v_H$).

V. DOWNWARD MOTION WITH DRAG AND GRAVITY

In this section we consider Case V, motion downwards with drag and gravity. In many respects much of the previous analysis is followed anew; however, with gravity now acting to accelerate the particle while the atmospheric drag tends to slow it down, the resulting motion has different features so that the analysis is not without interest.

We designate the downward velocity by $v_{\mathscr{D}}$ and define it by the dynamic equation in Equation (14):

$$dv_{\mathscr{D}}/dt = - C \exp(s_{\mathscr{D}}/H_o)\, v_{\mathscr{D}}^2 + g \tag{132a}$$

where

$$v_{\mathscr{D}} = ds_{\mathscr{D}}/dt \ , \qquad s_{\mathscr{D}} = h_o - h \tag{132b}$$

with the initial conditions

$$v_{\mathscr{D}} = v_o \ , \qquad s_{\mathscr{D}} = 0 \qquad \text{when } t = 0. \tag{132c}$$

The differential equation for $v_{\mathscr{D}}$ in Equation (132), differs from that for v_V in Equation (70a) by having $-s_V/H_o$ replaced by $s_{\mathscr{D}}/H_o$ and $- g$ by g. We are led by analogy to replace $-H_o$ by H_o and $-g$ by g, in the formula Equation (82a) for v_V, to write down by inspection a formula for $v_{\mathscr{D}}$:

$$v_{\mathscr{D}}^2 = v_o^2 \exp[2CH_o(1-w_{\mathscr{D}})] + 2gH_o \exp(-2CH_o\, w_{\mathscr{D}}) \int_1^{w_{\mathscr{D}}} \exp(2CH_o x)\, \frac{dx}{x} \tag{133a}$$

where

$$w_\ell = \exp(s_\ell/H_o) \ , \quad (dw_\ell/dt) = (w_\ell/H_o)\, v_\ell \tag{133b}$$

$$w_\ell = 1 \quad \text{when } s_\ell = 0 \ , \ t = 0 \qquad , \tag{133c}$$

We verify that Equation (133a) defines a velocity v_ℓ that satisfies the equation of motion by substitution.

Also since $s_\ell \to 1$ as $t \to 0$ it is obvious that $v_\ell \to v_o$ in Equation (133a), so that the initial condition on v_ℓ is satisfied.

Equation (133a) defines a formula for downward velocity, v_ℓ, of a particle, subjected to the combined action of drag and gravity, in terms of the descent distance s_ℓ.

To find the time t_ℓ of descent we can use either

$$dt = \frac{H_o}{w_\ell v_\ell}\, dw_\ell \qquad \text{from Equation (133b)}, \tag{134a}$$

or

$$dt = (ds_\ell/v_\ell), \tag{134b}$$

as definitions of an increment of time, dt, and by integration obtain the formulas

$$t_\infty = H_o \int_o^{\exp(s_\infty/H_o)} \frac{dw_\infty}{w_\infty v_\infty} \qquad (135a)$$

or

$$t_\infty = \int_o^{s_\infty} \frac{ds_\infty}{v_\infty} \qquad (135b)$$

The formulas in Equation (135) define the time t_∞ for the particle to descend a distance s_∞ against drag and with gravity; the function v_∞ is defined by Equation (133a).

When x is replaced by 1+y in the integral in Equation (133a), the formula for v_∞ becomes

$$v_\infty^{\ 2} = v_o^{\ 2} \exp[-2CH_o(w_\infty - 1)]\ G(w_\infty) \qquad (136a)$$

$$G(w_\infty) = 1 + (2gH_o/v_o^{\ 2}) \int_o^{w_\infty - 1} \exp(2CH_o y) \frac{dy}{1+y}, \quad w_\infty = \exp(s_\infty/H_o). \qquad (136b)$$

When $s_\infty \ll H_o$ the function w_∞ becomes

$$w_\infty = \exp(s_\infty/H_o) = 1 + (s_\infty/H_o) + \ldots \qquad (137)$$

so that

$$\int_o^{w_\infty - 1} \exp(2CH_o y) \frac{dy}{1+y} = \int_o^{s_\infty/H_o + \ldots} \exp(2CH_o y)[1-y + \ldots]\, dy \qquad (138)$$

If

$$\alpha = (1/2CH_o) \ , \ \xi_\infty = 2Cs_\infty, \quad \alpha\xi = (s_\infty/H_o) \ll 1$$

and

$$\lambda^2 = (2gH_o/v_o{}^2)\,\alpha(1 + \alpha) \tag{139}$$

then it can be shown that

$$\begin{aligned}(v_\infty/v_o)^2 &= \exp(-\xi_\infty) + \lambda^2\,[1-\exp(-\xi_\infty)] = \\ &\lambda^2 + (1 - \lambda^2)\exp(-\xi_\infty), \quad (s_\infty/H_o) \ll 1 \ ,\end{aligned} \tag{140}$$

That is to say

$$(v_\infty/v_o)^2 - \lambda^2 = (1 - \lambda^2)\exp(2Cs_\infty) \ , \qquad (s_\infty/H_o) \ll 1 \ , \tag{141}$$

Equation (141) defines a relationship between the downward velocity, v_∞, and the descent distance, s_∞, for a particle under the combined action of drag and gravity, $s_\infty \ll H_o$.

To find the time t_∞ when $s_\infty \ll H_o$, we set

$$u = \exp(Cs_\infty) \ , \qquad (du/dt) = C\,v_\infty u \tag{142}$$

so that

$$dt = \frac{1}{C}\frac{du}{uv_{\ell}} = \frac{1}{Cv_o}\frac{du}{\sqrt{\lambda^2u^2 + (1-\lambda^2)}} = \frac{1}{Cv_o\lambda}\frac{du}{\sqrt{(1-\lambda^2)/\lambda^2 + u^2}} \quad . \qquad (143)$$

When dt is integrated we obtain two formulas:

$$t_{\ell} = \frac{1}{Cv_o\lambda}\left[\sinh^{-1}\left(\frac{\lambda\exp(Cs_{\ell})}{\sqrt{1-\lambda^2}}\right) - \sinh^{-1}\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\right], \ \lambda^2 < 1, \ \frac{s_{\ell}}{H_o} \ll 1, \qquad (144a)$$

$$t_{\ell} = \frac{1}{Cv_o\lambda}\ \cosh^{-1}\left[\left(\frac{\lambda\exp(Cs_{\ell})}{\sqrt{\lambda^2-1}}\right) - \cosh^{-1}\left(\frac{\lambda}{\sqrt{\lambda^2-1}}\right)\right], \ \lambda^2 > 1, \ \frac{s_{\ell}}{H_o} \ll 1 \qquad (144b)$$

Equation (144) defines a formula for the time, t_{ℓ}, required for the particle to descend a distance s_{ℓ}, wherein $s_{\ell} \ll H_o$, when the downward motion of the particle is subjected to the restraint of drag and the acceleration of gravity.

Notice that Equation (141) can be expressed as

$$\exp(Cs_{\ell}) = \sqrt{\frac{1-\lambda^2}{(v_{\ell}/v_o)^2 - \lambda^2}} \quad \text{when } \lambda^2 < 1 \text{ and } \frac{v_{\ell}}{v_o} > \lambda, \ \frac{s_{\ell}}{H_o} \ll 1 \qquad (145a)$$

or as

$$\exp(Cs_{\ell}) = \sqrt{\frac{\lambda^2 - 1}{\lambda^2 - (v\ /v_o)^2}} \quad \text{when } \lambda^2 > 1 \text{ and } \frac{v_{\ell}}{v_o} < \lambda, \ \frac{s_{\ell}}{H_o} \ll 1 \ . \qquad (145b)$$

If $\alpha \ll 1$, then $\lambda^2 = (2gH_o/v_o^{\,2})\alpha = (g/Cv_o^{\,2})$, $2CH_o \gg 1$;

further, recall that

$$C = \frac{1}{2}\,(C_D A/m)\,\rho_o \exp(h_o/H_o) \text{ from Equation (10).}$$

The results in Equation (145) imply that when $s_\infty/H_o \ll 1$, if $\lambda^2 < 1$ then the minimum value of v_∞/v_o is λ; on the other hand, if $\lambda^2 > 1$, then v_∞/v_o can any value in the range $(0,1)$, i.e., $0 < v_\infty/v_o < 1$.

The equation in Equation (145) defines the descent distance, s_∞, in terms of the velocity ratio v_∞/v_o, provided $s_\infty/H_o \ll 1$.

An anomalous behavior of the velocity v_∞/v_o becomes apparent upon examination of Equation (141):

$$v_\infty/v_o = \lambda\sqrt{1 + ((1-\lambda^2)/\lambda^2)\exp(-2Cs_\infty)}\ , \quad (s_\infty/H_o) \ll 1; \qquad (147)$$

if

$$(|1-\lambda^2|/\lambda^2)\exp(-2Cs_\infty) \leqq 2f/\lambda\ , \text{ such that } 2f/\lambda \ll 1\ , \qquad (148)$$

then

$$v_\infty/v_o = \lambda + f + \ldots, \qquad \text{when } \exp(-2Cs_\infty) \leqq (2f\lambda/|1-\lambda^2|),\ \frac{s_\infty}{H_o} \ll 1. \qquad (149)$$

In other words

$$v = \sqrt{g/C} \;, \qquad \text{when } \frac{1}{C}\frac{\lambda}{\sqrt{1-\lambda^2}} < (s_\ell/H_o) \ll 1 \;, \tag{150}$$

The velocity of descent, v_ℓ, remains essentially at the constant value $\sqrt{g/C}$ for a finite segment of its descent, from an altitude h_o with an initial velocity v_o; the parameter C varies of course with h_o since $C = \frac{1}{2}\,(C_D A/m)\,\rho_o\,\exp(-h_o/H_o)$.

In this connection, if we assume that $dv_\ell/dt = 0$ for a given range of values s_ℓ, we can set dv_ℓ/dt in Equation (132), the dynamic equation of motion; we find that

$$v_\ell^2 = (g/C)\,\exp(-s_\ell/H_o) \;, \quad (dv_\ell/dt) = 0 \tag{151}$$

now if $s_\ell/H_o \ll 1$ then $\exp(-s_\ell/H_o) \cong 1$ and Equation (151) gives $v_\ell = \sqrt{g/C}$ in agreement with Equation (150). Indeed if one states that $dv_\ell/dt = -\delta'$ and that $\exp(s_\ell/H_o) = 1 - \delta''$, then

$$-\delta' = -C(1-\delta'')\,v_\ell^2 + g \;, \text{ so that, if } v_\ell = \sqrt{g/C} + \epsilon \;, \tag{152}$$

then

$$v_\ell - \sqrt{g/C} = \; = \sqrt{(g+\delta')/C(1-\delta'')} - \sqrt{g/C} = \sqrt{g/C}\left[\sqrt{\frac{1+\delta'/g}{1-\delta''}} - 1\right] \tag{153}$$

that is,

$$\frac{v_\infty - \sqrt{g/C}}{\sqrt{g/C}} = (1 + \tfrac{1}{2}\,\delta'/g)(1 - \tfrac{1}{2}\,\delta'') - 1 = \tfrac{1}{2}\left(\frac{\delta'}{g} - \delta''\right) \quad . \tag{154}$$

Equation (154) expresses the percent error in assuming the velocity, v_∞ a constant, $v_\infty = \sqrt{g/C}$, in terms of the variation, δ', in the rate of change of v_∞ and of the variation δ'' of the descent. As a closure to this discussion on the behavior of the particle for constant value of v_∞ one can look at the situation from a physical point of view: the change in potential energy of the particle goes into friction, into heating the atmosphere, by the frictional work done; that is to say, we have

$$mg(s_\infty - s_{\infty min}) = \int_{s_{\infty min}}^{s_\infty} m\,C\,\exp(s_\infty/H_o)\; v^2_{\infty const.}\; ds_\infty \tag{155}$$

so that

$$v^2_{\infty const} = \frac{g}{CH_o}\left[\frac{s_\infty - s_{\infty min}}{\exp(s_\infty/H_o) - \exp(s_\infty/H_{o_{min}})}\right] = \frac{g}{C}\;,\quad \exp(s_\infty/H_o) \ll 1 \tag{156}$$

agreeing with the previous values of $v_\infty =$ constant.

<u>Example 7</u>

We take the data of Examples 1, 3, and 5: $v_o = 2.5 \times 10^3$ cm/sec, $h_o = 71$ km, $C = 3.6 \times 10^{-4}$/cm, $H_o = 6.49 \times 10^5$ cm, $(1/\alpha) = 2CH_o = 467.2$,

$2gH_o/v_o^2 = 203.653$, $\alpha = .0021404$, $\lambda^2 = .43683$, $\lambda = .66093$, $(\lambda/\sqrt{1-\lambda^2}) =$ $.88072$, $(\lambda/\sqrt{1-\lambda^2}) = \sinh\,(.79447)$, $(1/Cv_{\infty}\lambda) = 1.681127$. In this instance, $\lambda^2 < 1$ so that we must choose $v_{\infty}/v_o > \lambda = .6609$; but this is not as revealing as assigning a value to t_{∞} in Equation (144a) and finding the corresponding value of s_{∞}. To this end we take $t_{\infty} = 10$ sec and solve for s_{∞}:

$$\sinh^{-1}\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\exp(Cs_{\infty})\right) = Cv_o\lambda\, t_{\infty} + \sinh^{-1}\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\right) = 6.74285 \qquad (157)$$

so that $\exp(Cs_{\infty}) = 480.24398 = \exp(13.08205)$:

hence

$$s_{\infty} = (13.08205/3.6 \times 10^{+4}) = 3.6339 \times 10^4 \text{ cm}, \quad (s_{\infty}/H_o) \doteq .05599 \ll 1. \qquad (158)$$

We infer that it is legitimate to use these approximation formulas; we conclude that in $t_{\infty} = 10$ sec, the particle will descend 3.6339×10^4 cm, assisted by gravity but opposed by aerodynamic drag. Compare this displacement with the distance the particle would go in 10 sec under gravity action alone, namely, 7.40308×10^4 cm. The speed v_{∞} at $t_{\infty} = 10$ sec is defined by Equation (141) wherein

$$(v_{\infty}/v_o)^2 = \lambda^2 + (1-\lambda^2)\exp(-2Cs_{\infty}) =$$

$$.4368314 + \left[.5631686/(480.244)^2\right] = .436831 + \ldots \qquad (159)$$

so that $(v_\infty/v_o) = \lambda = .66093$, $v_\infty = 1652.3$ cm/sec. Equation (159) indicates that v_∞/v_o is now insensitive to the value of $s_\infty \ll H_o$. Notice that if we take $s_{\infty_{min}} = 6{,}748.9$ cm, then $\exp(-Cs_{\infty_{min}}) = \exp(-2.4296) = (.8807196/10)$, so that $\exp(-2Cs_{\infty_{min}}) = \left[(.880719)^2/100\right] = (1/100)(\lambda^2/(1-\lambda^2)) = \left[(101/100)(\lambda^2/(1-\lambda^2))\right] - (\lambda^2/(1-\lambda^2))$, hence, by reference to Equation (141) where we have $\exp(-2Cs_\infty) = \left[(v_\infty/v_o)^2/(1-\lambda^2)\right] - (\lambda^2/(1-\lambda^2))$, we see that an error of less than 10% in v_∞/v_o occurs when $s_{\infty_{min}} = 6748.9$ cm is used instead of $s_\infty = 36.339 \text{ cm} \cong 5.3\, s_{\infty_{min}}$. A similar discussion reveals that $s_{\infty_{max}} = (1/10)H_o = 6.49 \times 10^4$ cm causes a variation in the value of v_∞/v_o of less than 10%. Hence, for s_∞ in the range $.0675 \text{ km} < s_\infty \leq .649$ km the variation of $v_\infty = 1652.3$ cm/sec is less than 10%. Equation (144a) is used to determine $t_{\infty_{min}}$, the time to descend to $s_{\infty_{min}} = 6748.9$ cm beginning at $h_o = 71$ km:

$$t_{\infty_{min}} = 1.0811272\left[\sinh^{-1}(10) - .79447\right] = 3.7048 \text{ sec} \tag{160}$$

since $(\lambda/\sqrt{1-\lambda^2})\exp(Cs_{\infty_{min}}) = 10$; further for $t_{\infty_{max}}$ corresponding to $s_{\infty_{max}} = 6.49 \times 10^4$ cm, with $(\lambda/\sqrt{1-\lambda^2})\exp(Cs_{\infty_{max}}) = .88072\exp(23.364) = \exp(23.237) = \sinh(23.930)$; hence

$$t_{\ell_{max}} = 1.681127\ (23.930 - .79447) = 38.894 \text{ sec} \qquad (161)$$

The time period of constant velocity $v_{\ell} = 1652.3$ cm/sec persists from $t_{\ell} = 3.705$ sec to $t_{\ell} = 38.894$ sec, or for a period of 35.189 sec; during 35.189 sec of descent at a constant speed 1652.3 cm/sec the particle falls a distance of 58,143 cm (notice that when we compute $s_{\ell_{max}} - s_{\ell_{min}}$ we obtain 64,900 - 6,749 = 58,151 cm!!).

PARTICLE	C_D	$\frac{m}{C_D A}$	$v_o\left(\frac{cm}{sec}\right)$	10^2	5×10^2	10^3	2.5×10^3	5×10^3	10^4	5×10^4	10^5	2.5×10^5	5×10^5
			s (km)	.0051	.026	.052	.13	.26	.512	2.56	5.2	12.79	25.58
I radius = 2μ = 2×10^{-4} cm, m = 1.17×10^{-10} gm	2	4.65×10^{-4} gm/cm^2	$\rho(h_o)\left(\frac{gm}{cm^3}\right)$	8.44×10^{-6}	1.69×10^{-6}	8.44×10^{-7}	3.38×10^{-7}	1.69×10^{-7}	8.44×10^{-8}	1.69×10^{-8}	8.44×10^{-9}	3.38×10^{-9}	1.69×10^{-9}
			h_o (km)	34.8	46.4	52	60.5	66	71	81	84.5	89	92.5
			C (1/cm)	.00907	.00182	.000907	.000363	.000182	9.07×10^{-5}	1.82×10^{-5}	9.07×10^{-6}	3.63×10^{-6}	1.8×10^{-6}
	1	9.36×10^{-4} gm/cm^2	$\rho(h_o)\left(\frac{gm}{cm^3}\right)$	1.64×10^{-5}	3.28×10^{-6}	1.64×10^{-6}	6.56×10^{-7}	3.28×10^{-7}	1.69×10^{-7}	3.38×10^{-8}	1.69×10^{-8}	6.76×10^{-9}	3.38×10^{-9}
			h_o (km)	30.6	41.6	46.6	54.5	61	66	77	81	86	89
			C (1/cm)	8.76×10^{-3}	1.75×10^{-3}	8.76×10^{-4}	3.50×10^{-4}	1.75×10^{-4}	9.00×10^{-5}	1.80×10^{-5}	9.00×10^{-6}	3.64×10^{-6}	1.80×10^{-6}
II radius = .5μ = 5×10^{-5} cm, m = 1.84×10^{-12}	2	1.2×10^{-4} gm/cm^2	$\rho(h_o)\left(\frac{gm}{cm^3}\right)$	2.16×10^{-6}	4.32×10^{-7}	2.16×10^{-7}	8.64×10^{-8}	4.32×10^{-8}	2.16×10^{-8}	4.32×10^{-9}	2.16×10^{-9}	8.64×10^{-10}	4.32×10^{-10}
			h_o (km)	44.4	58	64	71	75.5	80	88	91.5	95.5	99
			C (1/cm)	9.00×10^{-3}	1.80×10^{-3}	9.00×10^{-4}	3.6×10^{-4}	1.80×10^{-4}	9.00×10^{-5}	1.80×10^{-5}	9.00×10^{-6}	3.64×10^{-6}	1.80×10^{-6}
	1	2.4×10^{-4} gm/cm^2	$\rho(h_o)\left(\frac{gm}{cm^3}\right)$	4.32×10^{-6}	8.64×10^{-7}	4.32×10^{-7}	1.73×10^{-7}	8.64×10^{-8}	4.32×10^{-8}	8.64×10^{-9}	4.32×10^{-9}	1.73×10^{-9}	8.64×10^{-10}
			h_o (km)	39.4	52	58	66	71	75.5	84.5	88	92	95.5
			C (1/cm)	9.00×10^{-3}	1.80×10^{-3}	9.0×10^{-4}	3.60×10^{-4}	1.80×10^{-4}	9.0×10^{-5}	1.80×10^{-5}	9.0×10^{-6}	3.60×10^{-6}	1.80×10^{-6}
III radius = .03μ = 3×10^{-6} cm, m = 3.98×10^{-16} gm	2	7.04×10^{-6} gm/cm^2	$\rho(h_o)\left(\frac{gm}{cm^3}\right)$	1.27×10^{-7}	2.54×10^{-8}	1.27×10^{-8}	5.08×10^{-9}	2.54×10^{-9}	1.27×10^{-9}	2.54×10^{-10}	1.27×10^{-10}	5.08×10^{-11}	2.54×10^{-11}
			h_o (km)	68.5	79	82.5	87	90.5	94	102	106	111	115
			C (1/cm)	9.02×10^{-3}	1.80×10^{-3}	9.02×10^{-4}	3.60×10^{-4}	1.80×10^{-4}	9.02×10^{-5}	1.80×10^{-5}	9.02×10^{-6}	3.60×10^{-6}	1.80×10^{-6}
	1	1.41×10^{-5} gm/cm^2	$\rho(h_o)\left(\frac{gm}{cm^3}\right)$	2.54×10^{-7}	5.08×10^{-8}	2.54×10^{-8}	1.00×10^{-8}	5.08×10^{-9}	2.54×10^{-9}	5.04×10^{-10}	2.54×10^{-10}	1.02×10^{-10}	5.04×10^{-11}
			h_o (km)	62.5	74.5	79	83.5	87	90	98.5	102	107	110
			C (1/cm)	9.00×10^{-3}	1.80×10^{-3}	9.00×10^{-4}	3.54×10^{-4}	1.80×10^{-4}	9.00×10^{-5}	1.79×10^{-5}	9.00×10^{-6}	3.62×10^{-6}	1.79×10^{-6}

CASE I: Horizontal Motion with Drag; $t_H = 10$ sec., $v_H/v_o = 1/10$; $C = \frac{1}{2}\left(\frac{C_D A}{m}\right)\rho(h_o)$